Cubics

Jared Filseth

September 29, 2013

1 Introduction

In school curricula, much time is spent discussing quadratic equations. However, polynomials of higher orders (e.g. degree 3) are often highly undercovered. This should provide an introduction to cubics. A cubic function is a function of the form:

$$f(x) = ax^3 + bx^2 + cx + d$$

Often, problems involve setting this equal to zero; this then becomes a cubic equation:

$$ax^3 + bx^2 + cx + d = 0$$

2 The Cubic Formula

There is a general formula for finding the real root of a cubic, similar to the quadratic formula. This was first proven by Cardano in 1545. This might be worth memorizing, although it is somewhat cumbersome to use, so should probably be avoided whenever possible, if time is short:

Let

$$P = \frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}$$

$$Q = \frac{c}{3a} - \frac{b^2}{9a^2}$$

Then, 1 real root B of the equation is:

$$B = \sqrt[3]{P + \sqrt{P^2 + Q^3}} + \sqrt[3]{P - \sqrt{P^2 + Q^3}} - \frac{b}{3a}$$

The other 2 roots can be found by factoring out B and using the quadratic formula. All have at least 1 real root (easily proven by the Intermediate Value Theorem), so this will always work.

3 Smarter Techniques

3.1 Vieta's Formulas

Vieta's formulas are easily derived by expansion. Let p, q, and r be the 3 roots of the cubic equation. Multiplying, we get:

$$(x-p)(x-q)(x-r) = x^3 - (p+q+r)x^2 + (pq+qr+pr)x - pqr$$

Transferring this to a system of equations, we get that if $a = 1$, then:

$$b = -p - q - r \tag{1}$$

$$c = pq + qr + pr \tag{2}$$

$$d = -pqr \tag{3}$$

This can be very helpful, both in solving systems of equations and guessing roots, and has a lot of other applications.

3.2 Finding Integer/Rational Roots (when you highly suspect they exist)

3.2.1 The Rational Roots Theorem

This is a direct consequence of Vieta's formula (3). It says that all rational roots of a cubic equation take the form $\frac{p}{q}$, where p divides d, and q divides a. This is a very quick technique to apply, and helps narrow down your search significantly.

3.2.2 Plug and Chug

When you are dealing with small integer roots and coefficients, one technique is to plug the numbers obtained via the Rational Roots Theorem back into the equation and see if it evaluates to 0. For example, take the equation:

$$x^3 - 2x^2 + 2x - 4 = 0 \tag{4}$$

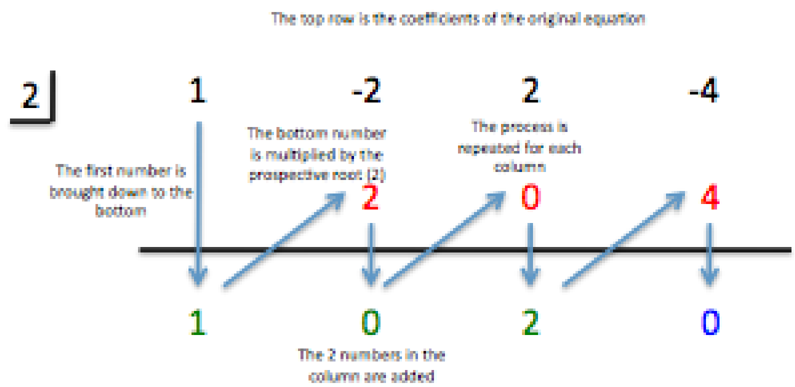
We want to see if $x = 2$ is a root of the polynomial, so we plug in:

$$2^3 - 2 * 2^2 + 2(2) - 4 = 8 - 8 + 4 - 4 = 0.$$

Thus, we know that x = 2 divides the polynomial, and we can use synthetic division (explained below) to factor out the $(x-2)$ term and find the resulting quadratic, from which the other roots are easily derived. However, when the cubic has large coefficients and/or potential roots, synthetic division is usually a better option.

3.2.3 Synthetic Division

Most of the time, this takes about as long as plug-and-chug (and often less time for larger numbers and non-integers). However, it has the advantage of allowing us to see the remaining factor after dividing the number. This is generally covered in school curricula. Here's how it works, using equation (4) and the prospective root of 2, as in the last example:



Since the remainder (the bottom right number, in blue) is 0, we know that $x-2$ divides 4, and that the remaining factor is x^2+0x+2, given by the coefficients in green.

Synthetic division also doubles as a way to quickly factor a known root out of a polynomial, to obtain the other factor. The Rational Roots Theorem, the plug-and-chug method, and synthetic division are all applicable to polynomials of degree higher than 3.

3.3 The Discriminant

The discriminant of a quadratic function ax^2+bx+c, as you probably know, is:

$$\Delta = b^2 - 4ac$$

The discriminant allows us to find the number and type of the roots in a quadratic, with these 3 cases:
$\Delta < 0$; The quadratic has no real roots, and 2 complex conjugate roots
$\Delta > 0$; The quadratic has 2 real roots, and no complex roots
$\Delta = 0$; The quadratic has 1 real root of multiplicity 2

The general cubic function ax^3+bx^2+cx+d also has a discriminant, which is given by:

$$\Delta = b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd$$

The discriminant of a cubic also allows us to find the number and type of roots:
$\Delta < 0$; The cubic has 1 real root, and 2 complex conjugate roots
$\Delta > 0$; The cubic has 3 real roots, and no complex roots
$\Delta = 0$; The cubic has 1 real root of multiplicity at least 2, and no complex roots

NOTE: All polynomials have discriminants. However, they quickly become very difficult to compute by hand; the general quartic equation has 16 terms in its discriminant, the quintic has 59 terms, the sextic has 246 terms, and the number continues to grow rapidly with degree.

4 Problems

1. Find the real root of the equation $8x^3 - 3x^2 - 3x - 1 = 0$. (AIME 2013). Try it without the cubic formula - there's a clever solution that isn't discussed anywhere above.

2. Prove that all cubics have at least 1 real root (do not use the discriminant).
3. The roots of the equation $x^3 + 78x + 666 = 0$ are p, q, and r. Find the value of $p^3 + q^3 + r^3$. Sidenote: Cubic equations with no x^2 term are called depressed cubic equations. These show up a lot on competitions, since they're generally much easier to work with than cubic equations with an x^2 term.
4. Extend Vieta's Formulas to quartic equations: Given the equation $ax^4+bx^3+cx^2+dx+e = 0$, and that the roots are p, q, r, and s, find expressions for b, c, d, and e in terms of a, p, q, r, and s.
5. Solve the following system of equations:

$$x + y + z = 17$$

$$xy + yz + xz = 94$$

$$xyz = 168$$

6. The product of two of the four zeros of the quartic equation

$$x^4 - 18x^3 + kx^2 + 200x - 1984 = 0$$

is -32. Find k. (USAMO 1984).

7. Prove Cardano's Cubic Formula (the one in Section 2) for a cubic equation of the form $x^3 + ax^2 + bx + c$, by eliminating the x^2 term using the substitution:

$$x = y - \frac{a}{3}$$

5 What Now?

Obviously, there's a lot of cool stuff about polynomials that this worksheet was unable to cover (even though it was really long compared to most). There's a lot of good stuff online, and I highly encourage you to do your own research!