Introduction to Induction

Jared Filseth

September 14, 2013

1 Introduction

Induction is a conceptually simple, but powerful proof technique. An inductive proof has (at least) two parts: a **base case**, which proves the statement for a small value, and an **inductive step**, which assumes the statement to be proven for a lower value (known as the **inductive hypothesis**), and uses it to prove the statement for a higher value, which in turn proves it for another value, etc., until all values specified in the problem are covered. This is somewhat akin to a line of dominoes-by knocking over the first domino, all are indirectly knocked over. It is one of the most common tactics in proof problems.

2 Example

Prove that for all positive integers n:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Base case:
Let $n = 1$. Then:

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2}$$

Inductive step:
Suppose this is true for n. Then:

$$\sum_{i=1}^{n+1} i = 1 + 2 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + n + 1$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}$$

Q.E.D.

3 Problems

1. Prove that for all positive integers n:

$$\sum_{i=1}^{n} n^3 = (\sum_{i=1}^{n} n)^2$$

2. For all positive integers n, let B_n denote the remaining figure after 1 square is removed from a 2^n by 2^n chessboard (not necessarily from the corner). Prove that for all n, B_n can be completely tesselated by copies of B_1.

3. Prove Fermat's Little Theorem: If a is an integer, p is a prime number, and a is not divisible by p, then:
$$a^{p-1} \equiv 1 \pmod{p}$$

4. We start with a pile of n stones. We divide the stones into two piles (however we wish), and write the product of the numbers of stones in the two piles on the blackboard. (For example, if $n = 25$, we might choose to divide the stones into piles of 11 and 14, in which case we would write 154 on the board.) We now choose one of the remaining piles, divide it into two smaller piles in any manner we choose, and again write the product of the numbers of stones in the two new piles on the blackboard. We repeat this process until we have n piles of 1 stone each. Prove that at the end of the process, no matter what choices we make along the way, the sum of the numbers written on the board will be $\binom{n}{2}$.

   (HINT: This proof requires **strong induction**, in which the inductive step assumes the inductive hypothesis for ALL integers less than n, not just $n - 1$).

5. A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n+2^{m_n+1}$, where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i. (USAMO)